

## FIVE REMARKS ABOUT RANDOM WALKS ON GROUPS

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**ABSTRACT.** The main aim of the present set of notes is to give new, short and essentially self-contained proofs of some classical, as well as more recent, results about random walks on groups. For instance, we shall see that the drift characterization of Liouville groups, due to Kaimanovich-Vershik and Karlsson-Ledrappier (and to Varopoulos in some important special cases) admits a very short and quite elementary proof. Furthermore, we give a new, and rather short proof of (a weak version of) an observation of Kaimanovich (as well as a small strengthening thereof) that the Poisson boundary of any symmetric measured group  $(G, \mu)$ , is doubly ergodic, and the diagonal  $G$ -action on its product is ergodic with unitary coefficients. We also offer a characterization of weak mixing for ergodic  $(G, \mu)$ -spaces parallel to the measure-preserving case.

We shed some new light on Nagaev's classical technique to prove central limit theorems for random walks on groups. In the interesting special case when the measured group admits a product current, we define a Besov space structure on the space of bounded harmonic functions with respect to which the associated convolution operator is quasicompact without any assumptions on finite exponential moments. For Gromov hyperbolic measured groups, this gives an alternative proof of the fact that every Hölder continuous function with zero integral with respect to the unique stationary probability measure on the Gromov boundary is a co-boundary.

Finally, we give a new and almost self-contained proof of a special case of a recent combinatorial result about piecewise syndeticity of product sets in groups by the author and A. Fish.

## 1. DRIFTS OF RANDOM WALKS AND THE LIOUVILLE PROPERTY

The study of random walks on countable groups is to a large extent concerned with the asymptotic behavior of convolution powers of some fixed probability measures on the groups. One is particularly interested in the growth of the integrals of certain geometrically defined functions against these convolution powers. For instance, let  $G$  be a countable group and let  $\mu$  be a probability measure on  $G$  with the property that the support of  $\mu$  generates  $G$  as a semigroup. We shall refer to  $(G, \mu)$  as a *measured group*. Given a left  $G$ -invariant and  $\mu$ -integrable metric  $d$  on  $G$ , we define the *drift* by

$$\ell_d(\mu) = \lim_n \frac{1}{n} \int_G d(g, e) d\mu^{*n}(g),$$

where  $\mu^{*n}$  denotes the  $n$ -th convolution power of  $\mu$ . An elementary sub-additivity argument (Fekete's Lemma) guarantees that the limit exists and is finite. From a probabilistic point of view, the drift  $\ell_d(\mu)$  measures the asymptotic linear speed (with respect to  $d$ ) of a sequence of products of independent and  $\mu$ -distributed elements in  $G$ . Since metrics are symmetric functions on  $G \times G$ , we always have  $\ell_d(\check{\mu}) = \ell_d(\mu)$ , where  $\check{\mu}(g) = \mu(g^{-1})$  for all  $g$  in  $G$ . We say that  $\mu$  is a *symmetric* probability measure (and  $(G, \mu)$  is a *symmetric measured group*) if  $\check{\mu} = \mu$ .

We note that if  $G$  is generated by a finite (symmetric) set  $S$ , then the word metric  $d_S$  with respect to  $S$  has the property that for every left  $G$ -invariant metric  $d$ , there exists a constant  $C$  such that the inequality  $d(g, e) \leq C \cdot d_S(g, e)$  holds for all  $g \in G$ , so in particular, if  $\ell_{d_S}(\mu) = 0$  for some probability measure  $\mu$ , then  $\ell_d(\mu) = 0$  as well.

Our aim in this section is to give a characterization of those finitely generated and symmetric measured groups  $(G, \mu)$  with  $\ell_d(\mu) = 0$  for *some* (and hence any) word-metric on  $G$  in terms of bounded left  $\mu$ -harmonic functions. Recall that a  $\mu$ -integrable complex-valued function  $f$  on  $G$  is *left*

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$\mu$ -harmonic if it satisfies

$$(\check{\mu} * f)(g) = \int_G f(sg) d\check{\mu}(s), \quad \forall g \in G,$$

and it is *right*  $\mu$ -harmonic if

$$(f * \mu)(g) = \int_G f(gs) d\mu(s), \quad \forall g \in G.$$

If  $f$  is both right *and* left  $\mu$ -harmonic, then we say that  $f$  is *bi- $\mu$ -harmonic*. Of special interest to us are the *bounded* left  $\mu$ -harmonic functions on  $G$ . Clearly, constant functions on  $G$  are left  $\mu$ -harmonic for every choice of  $\mu$ , and we say that  $(G, \mu)$  is (left) *Liouville* if there are no non-constant left  $\mu$ -harmonic functions. Since the function  $\check{f}(g) = f(g^{-1})$  is right  $\mu$ -harmonic if and only if  $f$  is left- $\mu$ -harmonic, we see that the notions of left and right Liouville coincide.

The original proof of the following theorem combined a series of fundamental observations of Avez [3], Derriennic [7], Kaimanovich-Vershik [17] and Karlsson-Ledrappier [20] respectively. In the special case when  $\mu$  is finitely supported, Varopoulos established this theorem in [26]. We shall give a short proof of the general theorem below.

**Theorem 1.1** (A geometric characterization of measured Liouville groups). *Let  $(G, \mu)$  be a finitely generated and symmetric measured group and suppose  $d$  is a word metric with respect to a finite symmetric generating set for  $G$ . Then  $\ell_d(\mu) = 0$  if and only if  $(G, \mu)$  is Liouville.*

**Remark 1.2.** Recall that a countable group  $G$  is *amenable* if every action of  $G$  by homeomorphisms on a compact hausdorff space  $X$  admits a  $G$ -invariant probability measure. It is not hard to see that finite groups and the group of integers are amenable, and that the class of amenable groups is closed under extensions and direct unions, which immediately shows that every finite extension of a solvable group is amenable. Furthermore, every finitely generated group of sub-exponential growth can be shown to be amenable, while free groups on at least two generators, and countable supergroups thereof are non-amenable.

Suppose  $(G, \mu)$  is a countable *non-amenable* measured group and let  $X$  denote a compact Hausdorff space, equipped with an action of  $G$  by homeomorphisms with no  $G$ -invariant probability measures. A simple application of Kaktuani's fixed point argument shows that there is always a probability measure  $\nu$  on  $X$  which satisfies the equation

$$\int_G \int_X \phi(g^{-1}x) d\nu(x) d\mu(g) = \int_X \phi(x) d\nu(x)$$

for all  $\phi \in C(X)$ . Since, by assumption,  $\nu$  cannot be  $G$ -invariant, there exists at least one  $\phi \in C(X)$  such that the function

$$f(g) = \int_X \phi(g^{-1}x) d\nu(x), \quad g \in G,$$

is *not* constant. It is readily verified that  $f$  is left  $\mu$ -harmonic and thus  $(G, \mu)$  is *not* Liouville. In particular, in view of Theorem 1.1, if  $G$  is a finitely generated *non-amenable* group, then  $\ell_d(\mu) > 0$  for *every* word-metric  $d$  on  $G$  and (symmetric) probability measure  $\mu$ .

**1.1. Liouville implies zero drift.** The proof of Theorem 1.1 splits naturally into two parts. The first one concerns the "if"-direction, for which the relevant result can be stated as follows.

**Theorem 1.3** (Karlsson-Ledrappier [20]). *If  $(G, \mu)$  is a measured Liouville group and  $d$  is a left invariant and  $\mu$ -integrable metric  $d$  on  $G$ , then there exists a real-valued and  $\mu$ -integrable homomorphism  $u$  on  $G$  such that*

$$\ell_d(\mu) = \int_G u(g) d\check{\mu}(g).$$

*In particular, if  $\mu$  is a symmetric probability measure on  $G$ , then  $\ell_d(\mu) = 0$  for every left invariant and  $\mu$ -integrable metric  $d$  on  $G$ .*

Let us now outline a simple proof of this theorem, which has some similarities with the arguments in the paper [8], where a more quantitative version of Theorem 1.3 is proved. We first note that by the triangle-inequality, the sequence

$$n \mapsto \int_G (d(g, x) - d(x, e)) d\mu^{*k}(x)$$

is bounded for every fixed  $g$  in  $G$ , so by a simple diagonal argument, there exists a sub-sequence  $(n_j)$  such that the limit

$$u(g) = \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \int_G (d(g, x) - d(x, e)) d\mu^{*k}(x)$$

exists for all  $g \in G$  and thus,

$$\int_G u(sg) d\check{\mu}(s) = u(g) + \int_G u(s) d\check{\mu}(s), \quad \forall g \in G.$$

We shall refer to functions  $u$  with this property as *left quasi- $\mu$ -harmonic*. By dominated convergence, we have

$$\begin{aligned} \int_G u(s) d\check{\mu}(s) &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \left( \int_G d(x, e) d\mu^{*(k+1)}(x) - \int_G d(x, e) d\mu^{*k}(x) \right) \\ &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \int_G d(x, e) d\mu^{*n_j}(x) = \ell_d(\mu). \end{aligned}$$

Furthermore, the triangle inequality guarantees that the function  $u$  is left Lipschitz, i.e. for every element  $g$  in  $G$ , we have

$$\sup_{x \in G} |u(xg) - u(x)| < +\infty.$$

The theorem of Karlsson-Ledrappier is now an immediate consequence of the following proposition, which is interesting in its own right.

**Proposition 1.1.** *If  $(G, \mu)$  is a Liouville measured group, then every left Lipschitz and left  $\mu$ -quasi-harmonic function on  $G$  which vanishes at the identity is a homomorphism.*

*Proof.* Note that if  $u$  is a left Lipschitz and left  $\mu$ -quasi-harmonic function on  $G$ , then for every  $g \in G$ , the function

$$v_g(x) = u(xg) - u(x), \quad x \in G,$$

is a *bounded* left  $\mu$ -harmonic function on  $G$ , and hence constant. Since  $u(e) = 0$ , we conclude that

$$u(xg) - u(x) = u(g)$$

for all  $g, x \in G$ , that is to say,  $u$  is a homomorphism.  $\square$

**Remark 1.4.** In particular, a Liouville group  $(G, \mu)$  without homomorphisms into the additive group of the real numbers cannot admit any non-constant *left Lipschitz* and *left (quasi)  $\mu$ -harmonic* functions. However, we shall see in Appendix I, that *every* countable symmetric measured group  $(G, \mu)$  always admits a non-constant *left Lipschitz* and *right  $\mu$ -harmonic* function.

**1.2. Zero drift implies Liouville.** We now tend to the proof of the "only if"-direction in Theorem 1.1, for which the relevant result can be stated as follows.

**Theorem 1.5** (Avez, Derriennic, Kaimanovich-Vershik, weaker form). *Let  $(G, \mu)$  be a finitely generated measured group and suppose  $d$  is a left-invariant word metric on  $G$  with respect to a finite symmetric generating set. If  $\ell_d(\mu) = 0$ , then  $(G, \mu)$  is Liouville.*

**Remark 1.6.** The classical route to this theorem employs the entropy theory of measured groups. One first shows that if  $\ell_d(\mu) < +\infty$  for some (and hence any) word metric  $d$  on  $G$ , then the limit

$$h(G, \mu) = \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{g \in G} \mu^{*n}(g) \cdot \log \mu^{*n}(g)$$

exists and is finite. We refer to  $h(G, \mu)$  as the (Avez) *entropy* of the measured group  $(G, \mu)$ , and the main result of Kaimanovich-Vershik in [17] asserts (under the assumption that  $\mu$  is symmetric and the Avez entropy is finite) that  $h(G, \mu) = 0$  if and only if  $(G, \mu)$  is Liouville. One is thus left with the task of showing that  $\ell_d(\mu) = 0$  implies  $h(G, \mu) = 0$ . This is taken care of what is sometimes referred to as the "fundamental" inequality (see e.g. Section 4 in [20]), which we now formulate. Let  $S$  be a finite and symmetric generating set for  $G$  and let  $d$  be the word metric associated to  $S$ . If one defines the *exponential volume growth* of  $(G, S)$  by

$$v(G, S) = \overline{\lim}_{n \rightarrow \infty} \frac{\log |S^n|}{n} \leq \log |S|$$

then

$$h(G, \mu) \leq v(G, S) \cdot \ell_d(\mu) = 0, \quad (1.1)$$

which finishes the (classical) proof of Theorem 1.5. Note that the argument gives a bit more, namely that if  $G$  has subexponential growth, i.e.  $v(G, S) = 0$ , then  $h(G, \mu) = 0$  and thus  $(G, \mu)$  is Liouville. By Theorem 1.3 and the remark following its statement, we conclude that  $G$  is amenable and  $\ell_d(\mu) = 0$ .

We shall now give a new alternative (and self-contained) proof of Theorem 1.5 which avoids the use of entropy theory. Let  $(G, \mu)$  be a measured group and denote by  $\mathcal{H}_1^\infty(G, \mu)$  the space of all bounded left  $\mu$ -harmonic functions on  $G$ . We say that a Borel probability space  $(X, \nu)$  is a  $(G, \mu)$ -*space* if  $X$  is equipped with an action of  $G$  by bi-measurable maps, which all preserve the measure class of  $\nu$ , such that

$$\int_G \int_X \phi(g^{-1}x) d\nu(x) d\mu(g) = \int_X \phi(x) d\nu(x)$$

for all  $\phi \in L^\infty(X, \nu)$ . Probability measures with this property are often referred to as  $\mu$ -*harmonic* (or  $\mu$ -*stationary*). Given a  $\mu$ -harmonic probability measure  $\nu$  on  $X$ , one readily checks that the association

$$P_\nu \phi(g) = \int_X \phi(g^{-1}x) d\nu(x), \quad g \in G.$$

defines an element in  $\mathcal{H}_1^\infty(G, \mu)$  for every  $\phi \in L^\infty(X, \nu)$ . We note that if  $\nu$  is  $G$ -invariant, then such elements are all constants. A remarkable fact, often attributed to Furstenberg, is that every measured group  $(G, \mu)$  admits a  $(G, \mu)$ -space  $(B, m)$ , which we shall refer to as the *Poisson boundary* of  $(G, \mu)$ , for which the linear map  $P_m$  above is in fact *isometric and onto*  $\mathcal{H}_1^\infty(G, \mu)$ . There are many constructions of the Poisson boundary of a measured group in the literature. We refer the reader to [9] for a detailed exposition of one of the more elementary constructions.

**Proposition 1.2** (Furstenberg). *For every measured group  $(G, \mu)$  there exists an ergodic  $(G, \mu)$ -space  $(B, m)$ , which we shall refer to as the Poisson boundary of  $(G, \mu)$ , such that the Poisson transform  $P : L^\infty(B, m) \rightarrow \mathcal{H}_1^\infty(G, \mu)$  defined by*

$$P\phi(g) = \int_B \phi(g^{-1}b) dm(b), \quad g \in G,$$

*is an isometric isomorphism. In particular,  $(B, m)$  is trivial if and only if  $m$  is  $G$ -invariant.*

**Remark 1.7.** As was pointed out by Jaworski in [16], the Poisson boundary  $(G, m)$  is strongly approximately transitive (SAT), i.e. for every measurable subset  $A \subset B$  of positive  $m$ -measure and for

every  $\varepsilon > 0$ , there exists  $g \in G$  such that  $m(gA) > 1 - \varepsilon$ . Indeed, since  $P$  is isometric, if  $A \subset B$  has positive  $m$ -measure, then

$$\sup_{g \in G} P\chi_A(g) = \sup_{g \in G} \int_B \chi_A(g^{-1}b) dm(b) = \sup_{g \in G} m(gA) = 1,$$

from which the SAT-property follows.

Since the  $G$ -action on  $(B, m)$  preserves the measure class of  $m$  (i.e. the set of all null-sets for  $m$ ), the Radon-Nikodym derivative

$$\sigma_m(g, b) = \frac{dg^{-1}m}{dm}(b)$$

is a well-defined non-negative element in  $L^1(B, m)$  for every  $g \in G$ , and one readily checks that

$$\sigma_m(g_1 g_2, b) = \sigma_m(g_1, b) \sigma_m(g_2, g_1 b) \quad (1.2)$$

for all  $g_1, g_2 \in G$  and for almost every  $b$  with respect to  $m$ , that is to say  $\sigma_m$  is a multiplicative cocycle for the  $G$ -action on  $(B, m)$ . A crucial feature with this cocycle is its  $\mu$ -harmonicity, namely

$$\int_G \sigma_m(g, b) d\mu^{*n}(g) = 1, \quad \text{for a.e. } b \text{ and for all } n \geq 1. \quad (1.3)$$

Indeed, for every  $\phi \in L^\infty(B, m)$ , one has

$$\int_B \left( \int_G \sigma_m(g, b) d\mu^{*n}(g) \right) \phi(b) dm(b) = \int_G \int_B \phi(g^{-1}b) dm(b) d\mu^{*n}(g) = \int_B \phi dm,$$

for all  $n$ , which immediately yields (1.3).

After a simple use of Jensen's inequality and (1.3), Theorem 1.5 can now be reformulated as follows.

**Theorem 1.8** (Kaimanovich-Vershik, weak version). *Let  $(G, \mu)$  be a finitely generated measured group with Poisson boundary  $(B, m)$  and suppose  $\ell_d(\mu) = 0$  for some (and hence any) word-metric with respect to a finite symmetric generating set. Then  $m$  is  $G$ -invariant, or equivalently*

$$\int_G \int_B \log \frac{dg^{-1}m}{dm}(b) dm(b) d\mu(g) = 0.$$

We note that equation (1.3) implies that  $\sigma_m(g, \cdot)$  is not only in  $L^1(B, m)$  for every  $g$ , but is in fact essentially bounded. This can be seen as follows. Since we assume that the support of  $\mu$  generates  $G$  as a semigroup, there exists for every  $s \in G$ , an integer  $n$  such that the measure  $\mu^{*n}(s)$  is positive, and thus

$$\sigma_m(s, b) \mu^{*n}(s) \leq \int_G \sigma_m(g, b) d\mu^{*n}(g) = 1$$

for  $m$ -almost every  $b$  in  $B$ . We conclude that

$$\|\sigma_m(s, \cdot)\|_\infty \leq \frac{1}{\mu^{*n}(s)} < +\infty,$$

where  $n$  is chosen as above, and

$$\sigma_m(s, \cdot) \geq \frac{1}{\sigma_m(s^{-1}, s \cdot)} \geq \frac{1}{\|\sigma_m(s^{-1}, \cdot)\|_\infty}, \quad \forall s \in G.$$

In particular, if we define

$$c_m(g, b) = -\log \sigma_m(g, b),$$

then one can think of  $c_m$  as a map from  $G$  into  $L^\infty(B, m)$ , which satisfies the equations

$$c_m(g_1 g_2) = c_m(g_1) + g_1^{-1} c_m(g_2), \quad \forall g_1, g_2 \in G,$$

where  $G$  acts on  $L^\infty(B, m)$  via the left regular representation. We shall refer to such maps  $c$  from  $G$  into  $L^\infty(B, m)$  as *cocycles*, and one readily checks that if  $c$  is a cocycle, then

$$\rho_c(g) = \|c(g)\|_\infty, \quad g \in G,$$

defines a semi-metric on  $G$ , i.e.

$$\rho_c(g_1, g_2) \leq \rho_c(g_1) + \rho_c(g_2), \quad \forall g_1, g_2 \in G.$$

We reserve the notation  $\rho_m$  for the semi-metric associated to the cocycle  $c_m$  above and refer to  $\rho_m$  as the *canonical semi-metric on  $G$  associated with  $(G, \mu)$* .

**Remark 1.9.** In order to get a better feeling for the canonical semi-metric of a measured group, let us consider the case when  $G = \mathbb{F}_2$ , the free group on two free generators  $a$  and  $b$ , equipped with the (symmetric) probability measure

$$\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).$$

Let  $\partial\mathbb{F}_2$  denote the compact space of all infinite one-sided reduced words in  $a$  and  $b$  and their inverses and note that the action of  $\mathbb{F}_2$  on itself extends to an action by homeomorphisms on  $\partial\mathbb{F}_2$ . One readily checks that the Borel probability measure  $m$  on  $\partial\mathbb{F}_2$  which assigns the same measure to all cylinder sets in  $\partial\mathbb{F}_2$  corresponding to words of the same word length is  $\mu$ -harmonic, and one can prove that  $(\partial\mathbb{F}_2, m)$  realizes the Poisson boundary for  $(\mathbb{F}_2, \mu)$ . Furthermore, the Radon-Nikodym cocycle of  $m$  is given by

$$\sigma_m(g, \xi) = 3^{-(\|g\| - 2(g, \xi))}, \quad (g, \xi) \in \mathbb{F}_2 \times \partial\mathbb{F}_2,$$

where  $\|\cdot\|$  denotes the word metric on  $\mathbb{F}_2$  (with respect to  $a$  and  $b$  and their inverses) and  $(g, \xi)$  is the length of the longest common sub-word of  $g$  and  $\xi$  (this is often referred to as the *confluent* or *Gromov product* in the literature). A straightforward calculation now yields

$$\rho_m(g) = \|g\| \cdot \ln 3, \quad \forall g \in G,$$

which in particular shows that in this special case,  $\rho_m$  is in fact a metric.

The following simple proposition relates the asymptotic behavior of a cocycle  $c$  to the vanishing of the drift of  $(G, \mu)$  with respect to word-metrics on finitely generated groups, and finishes the proof of our version of the theorem of Kaimanovich-Vershik.

Recall that a  $\mu$ -harmonic mean  $\lambda$  on  $L^\infty(B, m)$  is a functional on  $L^\infty(B, m)$  which is positive, i.e. gives non-negative values to non-negative elements in  $L^\infty(B, m)$ , normalized, i.e.  $\lambda(1) = 1$  and satisfies

$$\int_G \lambda(g \cdot \phi) d\mu(g) = \lambda(\phi), \quad \forall \phi \in L^\infty(B, m)$$

where  $G$  acts on  $L^\infty(B, m)$  via the left regular representation. In particular, the measure  $m$  is a  $\mu$ -harmonic mean on  $L^\infty(B, m)$ .

**Proposition 1.3.** *Let  $(G, \mu)$  be a finitely generated measured group such that  $\ell_d(\mu) = 0$  for some left invariant word-metric  $d$  with respect to a finite symmetric generating set of  $G$ . Let  $(B, m)$  denote the Poisson boundary of  $(G, \mu)$ . If  $c : G \rightarrow L^\infty(B, m)$  is a cocycle, then*

$$\int_G \lambda(c(g)) d\mu(g) = 0$$

for every  $\mu$ -harmonic mean  $\lambda$  on  $L^\infty(B, m)$ . In particular, we have

$$\int_B \int_G \log \frac{dg^{-1}m}{dm}(b) d\mu(g) dm(b) = 0,$$

so  $m$  is  $G$ -invariant, and thus  $(B, m)$  is trivial.

*Proof.* Since  $G$  is finitely generated, there exists for every cocycle  $c : G \rightarrow L^\infty(B, m)$  a constant  $C_c$  such that

$$\rho_c(g) \leq C_c \cdot d(g, e), \quad \forall g \in G,$$

where  $d$  is a word-metric on  $G$  with respect to a finite symmetric generating set of  $G$ . We assume that  $\ell_d(\mu) = 0$ , and thus

$$\lim_n \frac{1}{n} \int_G \rho_c(g) d\mu^{*n}(g) = 0$$

for every cocycle  $c$ . If  $\lambda$  is a  $\mu$ -harmonic mean on  $L^\infty(B, m)$ , one readily checks that

$$\int_G \lambda(c(g)) d\mu^{*n}(g) = n \cdot \int_G \lambda(c(g)) d\mu(g)$$

for all  $n$ , and thus

$$\left| \int_G \lambda(c(g)) d\mu(g) \right| = \left| \frac{1}{n} \int_G \lambda(c(g)) d\mu^{*n}(g) \right| \leq \frac{1}{n} \int_G \|c(g)\|_\infty d\mu^{*n}(g) \rightarrow 0,$$

which finishes the proof.  $\square$

## 2. ERGODICITY WITH UNITARY COEFFICIENTS

We now turn to some ergodic-theoretical aspects of random walks on groups. As we have seen, to every measured group  $(G, \mu)$  one can associate an ergodic  $(G, \mu)$ -space  $(B, m)$ , called the Poisson boundary of  $(G, \mu)$  with the remarkable property that the linear map  $P : L^\infty(B, m) \rightarrow \mathcal{H}_1^\infty(G, \mu)$  defined by

$$P\phi(g) = \int_B \phi(g^{-1}b) dm(b), \quad g \in G,$$

is an isometric isomorphism. The aim of this section is to give a short proof of (a weak version) of an observation of Kaimanovich in [19], that the mere fact that this is an isomorphism onto  $\mathcal{H}_1^\infty(G, \mu)$  automatically forces significantly stronger ergodicity properties.

**Theorem 2.1** (Kaimanovich, weak version). *Let  $(G, \mu)$  be a measured group and denote by  $(B, m)$  and  $(\check{B}, \check{m})$  the Poisson boundaries of  $(G, \mu)$  and  $(G, \check{\mu})$  respectively. If  $(Y, \eta)$  is any ergodic probability measure preserving  $G$ -space, then the diagonal action on the triple  $(B \times \check{B} \times Y, m \otimes \check{m} \otimes \eta)$  is ergodic.*

**Remark 2.2.** It is not hard to show (see for instance the recent survey by Glasner-Weiss [13]) that the theorem above can be equivalently formulated as follows: For every unitary  $G$ -representation  $(\mathcal{H}, \pi)$  on a Hilbert space  $\mathcal{H}$ , any measurable  $G$ -equivariant map  $F : B \times \check{B} \rightarrow \mathcal{H}$  must be essentially constant. Kaimanovich proves in [19] the a priori stronger statement that one can assert the same thing about  $G$ -equivariant and weak\*-measurable maps from  $B \times \check{B}$  into *any* isometric  $G$ -representation on any (separable) Banach space. For certain applications in bounded cohomology, this seemingly stronger statement is needed. Since we wish to keep the discussions in this paper fairly short, we shall confine ourselves to the setting of Theorem 2.1, although many of the techniques we shall describe can be used to give a complete proof of the main result in [19].

To start the proof of Theorem 2.1, we first observe that if  $(\check{B}, \check{m})$  is the Poisson boundary of  $(G, \check{\mu})$  and  $(Y, \eta)$  is any probability measure preserving  $G$ -space, then the diagonal  $G$ -action on  $(\check{B} \times Y, m \otimes \eta)$  is a  $(G, \check{\mu})$ -space. Hence Theorem 2.1 follows immediately from the following result, which we have not been able to directly locate in the literature.

**Theorem 2.3.** *Let  $(G, \mu)$  be a measured group with Poisson boundary  $(B, m)$  and suppose  $(X, \nu)$  is an ergodic  $(G, \check{\mu})$ -space. Then the diagonal  $G$ -action on the product space  $(B \times X, m \otimes \nu)$  is ergodic.*

**Remark 2.4.** In order to see how Theorem 2.1 follows this statement, we argue in two steps. First note that if  $(Y, \eta)$  is an ergodic probability measure preserving  $G$ -space, then it is an ergodic  $(G, \check{\mu})$ -space as

well, and Theorem 2.3 implies that  $X = \check{B} \times Y$ , with the probability measure  $\nu = \check{m} \otimes \eta$ , is an *ergodic*  $(G, \check{\mu})$ -space. If we now apply Theorem 2.3 to the diagonal  $G$ -action on the direct product

$$(B \times X, m \otimes \nu) = (B \times \check{B} \times Y, m \otimes \check{m} \otimes \eta),$$

then we conclude that it is also ergodic, which is exactly the assertion of Theorem 2.1.

In the case when  $(X, \nu)$  is an ergodic probability measure preserving  $G$ -space, Theorem 2.3 is due to Aaronson and Lemańczyk in [1]. Their proof however follows quite different lines than ours.

We now begin our proof of Theorem 2.3. Let  $(X, \mu)$  be an ergodic  $(G, \check{\mu})$ -space, and suppose  $f$  is a  $G$ -invariant essentially bounded function on  $B \times X$ , which we may without loss of generality assume to have zero integral. We wish to prove that  $f$  vanishes identically.

For this purpose, we shall show that

$$\int_B f(b, x) \phi(b) dm(b) = 0, \quad \text{for } \nu\text{-a.e. } x \text{ in } X \quad (2.1)$$

for all  $\phi \in L^1(B, m)$ , and thus

$$\int_X \int_B f(b, x) \phi(b) \psi(x) dm(b) d\nu(x) = 0$$

for all  $\phi \in L^1(B, m)$  and  $\psi \in L^1(X, \nu)$ , which shows that  $f$  must vanish identically, establishing ergodicity for the diagonal action on  $(B \times X, m \otimes \nu)$ . To prove (2.1), we argue as follows. Define

$$s(x) = \int_B f(b, x) dm(b)$$

and note that

$$\begin{aligned} \int_G s(gx) d\mu(g) &= \int_G \int_B f(b, gx) dm(b) d\mu(g) \\ &= \int_G \int_B f(g^{-1}b, x) dm(b) d\mu(g) \\ &= \int_B f(b, x) dm(b) = s(x), \end{aligned}$$

since  $m$  is  $\mu$ -harmonic. The following lemma now shows that  $s$  must vanish almost everywhere with respect to the measure  $\nu$  (recall that  $(X, \nu)$  is an ergodic  $(G, \check{\mu})$ -space).

**Lemma 2.1.** *Let  $(X, \nu)$  be a  $(G, \mu)$ -space and suppose that  $s \in L^\infty(X, \nu)$  satisfies the equation*

$$s = \int_G s(g^{-1}\cdot) d\mu(g) \quad \text{in } L^\infty(X, \nu).$$

*Then  $f$  is essentially  $G$ -invariant. In particular, if  $(X, \nu)$  is ergodic, then  $f$  equals its  $\nu$ -integral almost everywhere.*

*Proof.* We may assume that  $s$  is real-valued. Since the support of  $\mu$  is assumed to generate  $G$ , it suffices to show that

$$\int_G \int_X |s(g^{-1}x) - s(x)|^2 d\mu^{*k}(g) d\nu(x) = 0$$

for all  $k$ . However, upon expanding the square, and using the harmonicity of  $\nu$ , we see that

$$\int_G \int_X |s(g^{-1}x) - s(x)|^2 d\mu^{*k}(g) d\nu(x) = 2 \cdot \left( \int_X |s(x)|^2 d\nu(x) - \int_X s(x) \left( \int_G s(g^{-1}x) d\mu^{*k}(g) \right) d\nu(x) \right),$$

which clearly vanishes by our assumption on  $s$ .  $\square$



Going back to the proof of Theorem 2.3, we can now conclude that

$$\int_B f(b, x) d\mathbf{m}(b) = 0 \quad \text{for } \nu\text{-a.e. } x \text{ in } X,$$

and thus

$$\int_B f(b, gx) d\mathbf{m}(b) = \int_B f(g^{-1}b, x) d\mathbf{m}(b) = \int_B f(b, x) \sigma_{\mathbf{m}}(g, b) d\mathbf{m}(b) = 0,$$

for almost every  $x$  in  $X$  and for all  $g$  in  $G$ . In particular, we have

$$\int_B f(b, x) \phi(b) d\mathbf{m}(b) = 0, \quad \text{for } \nu\text{-a.e. } x \text{ in } X,$$

and for all  $\phi$  in the linear span of all  $\sigma_{\mathbf{m}}(g, \cdot)$  as  $g$  ranges over  $G$ . Hence Theorem 2.3 will follow from the following simple lemma, which is essentially just a reformulation of Proposition 1.2.

**Lemma 2.2.** *Let  $(G, \mu)$  be a measured group and denote by  $(B, \mathbf{m})$  its Poisson boundary. Then the linear span*

$$\mathcal{R}_{\mathbf{m}} = \text{span} \left\{ \frac{dg^{-1}\mathbf{m}}{d\mathbf{m}} : g \in G \right\} \subset L^1(B, \mathbf{m})$$

*is dense in  $L^1(B, \mathbf{m})$ .*

*Proof.* If this span would not be dense, then by Hahn-Banach's Theorem, there exists a non-zero functional  $\phi \in L^1(B, \mathbf{m})^* = L^\infty(B, \mathbf{m})$  such that

$$\int_X \phi(b) \frac{dg^{-1}\mathbf{m}}{d\mathbf{m}}(b) d\mathbf{m}(b) = \int_X \phi(g^{-1}b) d\mathbf{m}(b) = 0, \quad \forall g \in G,$$

or equivalently,  $P\phi(g) = 0$ , where  $P$  is as in Proposition 1.2. Since  $P$  is an isomorphism, we conclude that  $\phi$  vanishes identically, which is a contradiction.  $\square$

We finish this section with yet another consequence of Proposition 1.2 which seems to be rarely stressed in the literature. It was first observed by Kaimanovich in [18], but the analogous case (in fact, concerning positive bi-harmonic functions) for amenable *connected* measured group goes back to Raugi in [25]).

**Corollary 2.1** (Choquet-Deny, Blackwell, Kaimanovich). *For every probability measure  $\mu$  on a countable group  $G$ , there are no non-constant bounded functions which are both left and right  $\mu$ -harmonic. In particular, measured abelian groups do not admit any non-constant bounded harmonic functions.*

**Remark 2.5.** The last assertion is immediate if  $\mu$  is symmetric. If it is not and  $\phi$  is a bounded left  $\mu$ -harmonic function with  $\phi(e) = 0$  (and hence right  $\check{\mu}$ -harmonic), then the symmetrized function

$$\psi(g) = \phi(g) + \phi(g^{-1}), \quad g \in G,$$

is a bounded left and right  $\mu$ -harmonic and thus identically zero by the corollary above. This forces the identities  $\phi(g) = -\phi(g^{-1})$  for all  $g \in G$ , and thus  $\phi$  is both left and right  $\mu$ -harmonic, and hence constant.

*Proof.* Let  $(B, \mathbf{m})$  denote the Poisson boundary of  $(G, \mu)$  and suppose  $f$  is a bounded left and right  $\mu$ -harmonic function on  $G$ . Let  $\phi$  denote the unique element in  $L^\infty(B, \mathbf{m})$  such that  $f = P\phi$ , where  $P$  is the linear map defined in Proposition 1.2. Note that the uniqueness of  $\phi$  forces the identity

$$\int_X \phi(g^{-1}\cdot) d\mu(g) = \phi \quad \text{in } L^\infty(B, \mathbf{m}).$$

By the last lemma, we conclude that  $\phi$  is  $G$ -invariant and thus constant, since  $(B, \mathbf{m})$  is ergodic.  $\square$

### 3. WEAK MIXING FOR $(G, \mu)$ -SPACES

The aim of this section is to characterize weakly mixing  $(G, \mu)$ -spaces as exactly those which do not admit any non-trivial probability measure preserving factors with discrete spectrum. The author has not been able to locate an explicit formulation of this equivalence in the literature, although it should be stressed that the characterization does follow from applying a series of classical and well-known techniques combined with the fact that WAP-actions are  $\mu$ -stiff in the sense of Furstenberg, which was established in [10]. However, as the proof of this fact utilizes some serious machinery from the theory of Ellis semigroups and weakly almost periodic functions, the route to the characterization of weakly mixing  $(G, \mu)$ -spaces (following these lines) is not very direct. We shall try to outline below a more direct approach.

First recall that a non-singular  $G$ -space  $(X, \nu)$  is *weakly mixing* if for every ergodic probability measure preserving  $G$ -space  $(Y, \eta)$ , the diagonal  $G$ -action on  $(X \times Y, \nu \otimes \eta)$  is ergodic. If  $\nu$  is  $G$ -invariant, then this is equivalent to the absence of a non-trivial factor with discrete spectrum, that is to say, a probability measure preserving  $G$ -space  $(Z, \xi)$  with the property that the corresponding unitary (Koopman) representation on  $L^2(Z, \xi)$  decomposes into a *direct sum* of finite dimensional subrepresentations. By a classical theorem of Mackey in [22], an ergodic probability measure preserving  $G$ -space with discrete spectrum is very special. Indeed, it is always isomorphic to an isometric  $G$ -action on a compact homogeneous space, that is to say, there exists a compact group  $K$  and a closed subgroup  $K_0 < K$  and a homomorphism  $\tau : G \rightarrow K$  with dense image such that the  $\tau(G)$ -action on  $K/K_0$  (with the Haar probability measure) is isomorphic (as a  $G$ -space) to  $(Z, \xi)$ . In particular, if  $G$  is a minimally almost periodic group (which means that there are no non-trivial finite dimensional unitary  $G$ -representations whatsoever), then every ergodic probability measure preserving  $G$ -space is automatically weakly mixing.

It is not true in general that an ergodic non-weakly mixing non-singular  $G$ -space admits a probability measure preserving factor with discrete spectrum. In fact, Aaronson and Nadkarni constructs in [2] a probability measure on a compact group, which is non-singular and ergodic with respect to dense cyclic subgroup, so that the corresponding ergodic non-singular  $\mathbb{Z}$ -space (which is certainly not weakly mixing) does not admit any non-trivial probability measure preserving factors whatsoever.

However, in the category of  $(G, \mu)$ -spaces the situation is much nicer and the aim of this section is to give a self-contained proof of the following theorem, which is not new and certainly known to experts.

**Theorem 3.1** (Characterization of weak mixing). *Let  $(G, \mu)$  be a measured group and suppose  $(X, \nu)$  is an ergodic  $(G, \mu)$ -space. Then  $(X, \nu)$  is weakly mixing if and only if  $(X, \nu)$  does not admit a non-trivial probability measure preserving factor with discrete spectrum. In particular, if  $G$  is minimally almost periodic, then every ergodic  $(G, \mu)$ -space is weakly mixing.*

**Remark 3.2.** In particular this theorem applies to the Poisson boundary of  $(G, \mu)$ , which certainly does not have any probability measure preserving factors whatsoever, and thereby giving yet another proof of the weak mixing of Poisson boundaries, originally due to Aaronson and Lemańczyk in [1]. Note however that the theorem does *not* directly apply to the setting of Theorem 2.1 since products of Poisson boundaries are not  $(G, \mu)$ -spaces in general (unless of course, they are trivial).

Let us now begin the proof of Theorem 3.1, which naturally falls into two steps, both of which are essentially classical, and only the first step needs to be complemented with a less classical argument concerning probability measure preserving factors. As we have already mentioned above, this argument could be replaced by a nice, but not very elementary observation of Furstenberg and Glasner in [10] about  $\mu$ -harmonic measures on WAP-spaces. However, since no self-contained proof of Theorem 3.1 seems to exist in the literature, it makes sense to outline a more direct route in this paper, and to collect here all the necessary arguments, although we do allow ourselves to be a bit sketchy in the more classical arguments.

For the first step, we let  $(G, \mu)$  be a measured group and  $(X, \nu)$  is an ergodic  $(G, \mu)$ -space. Suppose there exists an ergodic probability measure preserving  $G$ -space  $(Y, \eta)$  such that the diagonal  $G$ -action on  $(X \times Y, \nu \otimes \eta)$  is *not* ergodic, that is to say, there exists a non-constant essentially bounded real-valued function  $f$  on  $X \times Y$ . Without loss of generality, we can assume that  $f$  is bounded by one, so that the map

$$\pi_f : X \rightarrow B_1(L^2(Y, \eta))$$

given by  $\pi_f(x) = f(x, \cdot) \in B_1(L^2(Y, \eta))$  is well-defined for almost every  $x$  in  $X$ , where  $B_1(L^2(Y, \eta))$  denotes the unit ball in the Hilbert space  $L^2(Y, \eta)$ . Since  $f$  is assumed to be  $G$ -invariant, one can readily verify that  $\pi_f$  is a (weakly measurable) factor map from  $X$  into the  $(G, \mu)$ -space  $(B_1(L^2(Y, \eta)), \pi_*\nu)$ , where  $G$  acts on the unit ball  $B_1(L^2(Y, \eta))$  via the (unitary) Koopman operator on  $L^2(Y, \eta)$  (recall that  $(Y, \eta)$  is measure-preserving). We note that since  $f$  is non-constant, the corresponding factor is non-trivial.

More generally, suppose  $(\mathcal{H}, \pi)$  is a unitary  $G$ -representation on a Hilbert space  $\mathcal{H}$ . Then  $\pi$  induces a (weakly continuous) action of  $G$  on the unit ball  $B_1(\mathcal{H})$ , and if  $\nu$  is a  $\mu$ -harmonic probability measure (with respect to this  $G$ -action) on  $B_1(\mathcal{H})$ , then we shall refer to  $(B_1(\mathcal{H}), \nu)$  as a *Hilbertian  $(G, \mu)$ -space*. Theorem 3.1 will then follow from the following proposition.

**Proposition 3.1.** *Every Hilbertian  $(G, \mu)$ -space is measure-preserving and has discrete spectrum.*

We begin by proving that Hilbertian  $(G, \mu)$ -spaces are measure-preserving. To do so, we first note that by Stone-Weierstrass Theorem, the linear span of the constants and all functions of the form

$$\phi(x) = \langle y_1, x \rangle \cdots \langle y_k, x \rangle, \quad x \in B_1(\mathcal{H}), \quad y_1, \dots, y_k \in \mathcal{H} \quad (3.1)$$

is dense in  $C(B_1(\mathcal{H}))$ , when  $B_1(\mathcal{H})$  is equipped with the weak topology, and we wish to prove that

$$\int_X \phi(g^{-1}x) d\nu(x) = \int_X \phi(x) d\nu(x), \quad \forall g \in G, \quad (3.2)$$

for all  $y_1, \dots, y_k \in \mathcal{H}$ . Since

$$\begin{aligned} \int_G \int_{B_1(\mathcal{H})} \phi(g^{-1}x) d\nu(x) d\mu(g) &= \left\langle y_1 \otimes \cdots \otimes y_k, \int_G \pi^{\otimes k}(g) \xi_\nu d\mu(g) \right\rangle, \\ &= \left\langle y_1 \otimes \cdots \otimes y_k, \xi_\nu \right\rangle, \end{aligned}$$

for all  $y_1, \dots, y_k \in \mathcal{H}$ , where  $\pi^{\otimes k}$  denotes the  $k$ -th tensor product representation of  $(\mathcal{H}, \pi)$ , and

$$\xi_\nu = \int_X x \otimes \cdots \otimes x d\nu(x),$$

we can conclude that

$$\pi^{\otimes k}(\mu) \xi_\nu = \int_G \pi^{\otimes k}(g) \xi_\nu d\mu(g) = \xi_\nu.$$

Hence (3.2) will follow from the following simple lemma (applied to all finite tensor product representations of  $(\mathcal{H}, \pi)$ ).

**Lemma 3.1.** *Let  $(\mathcal{H}, \pi)$  be a unitary  $G$ -representation and suppose  $\xi \in \mathcal{H}$  satisfies  $\pi(\mu)\xi = \xi$ . Then  $\xi$  is  $\pi(G)$ -invariant.*

*Proof.* We may without loss of generality assume that  $\|\xi\| = 1$ . Since  $\pi$  is unitary, the equation  $\pi(\mu)\xi = \xi$  simply means that a convex average of points of the form  $\pi(g)\xi$ , for  $g$  in the support of  $\mu$ , equals  $\xi$ . However, by the strict convexity of the unit ball in  $\mathcal{H}$ , this can only happen if  $\pi(g)\xi = \xi$  for all  $g$  in the support of  $\mu$ . Since the support of  $\mu$  is assumed to generate  $G$ , we conclude that  $\xi$  is  $\pi(G)$ -invariant.  $\square$

It remains to show that the measure-preserving  $G$ -space  $(B_1(\mathcal{H}), \nu)$  has discrete spectrum. For this purpose, we define the closed linear subspace

$$\mathcal{H}_o = \overline{\left\{ \nu \in \mathcal{H} : \text{the cyclic span of } \nu \text{ is finite-dimensional} \right\}} \subset \mathcal{H}.$$

One readily checks that  $\mathcal{H}_0$  is a sub-representation of  $\mathcal{H}$ , and thus its orthogonal complement  $\mathcal{H}_1$  is a sub-representation with the property that it does not have any finite-dimensional sub-representations whatsoever. Furthermore, we have

$$B_1(\mathcal{H}) = \left\{ (\xi, \eta) \in \mathcal{H}_0 \oplus \mathcal{H}_1 : \|\xi\|_0^2 + \|\eta\|_1^2 \leq 1 \right\} \subset B_1(\mathcal{H}_0) \times B_1(\mathcal{H}_1).$$

We have canonical continuous  $G$ -equivariant projections  $\pi_0$  and  $\pi_1$  from  $B_1(\mathcal{H})$  onto  $B_1(\mathcal{H}_0)$  and  $B_1(\mathcal{H}_1)$  respectively, and it is not hard to show that the  $G$ -space  $(B_1(\mathcal{H}_0), \pi_0)_* \nu$  has discrete spectrum. Hence it suffices to show the following lemma.

**Lemma 3.2.** *Suppose  $(\mathcal{H}, \pi)$  is a unitary  $G$ -representation with no (non-trivial) finite-dimensional sub-representations. If  $\nu$  is a  $G$ -invariant probability measure on  $B_1(\mathcal{H})$ , then it is concentrated at zero.*

*Sketch of proof.* Since  $C(B_1(\mathcal{H}))$  is generated by limits of linear combinations of products of the form as in (3.1), it suffices to show that

$$\int_{B_1(\mathcal{H})} |\langle y, x \rangle|^2 d\nu(x) = 0, \quad \forall y \in \mathcal{H}. \quad (3.3)$$

In order to establish (3.3), we note that

$$\int_{B_1(\mathcal{H})} |\langle y, x \rangle|^2 d\nu(x) = \langle y \otimes y^*, \int_{B_1(\mathcal{H})} x \otimes x^* d\nu(x) \rangle, \quad \forall y \in \mathcal{H},$$

where the  $*$  refers to complex conjugation, and since  $\nu$  is  $G$ -invariant, the vector

$$\xi = \int_{B_1(\mathcal{H})} (x \otimes x^*) d\nu(x) \in \mathcal{H} \otimes \mathcal{H}^*$$

is invariant under  $\pi \otimes \pi^*(G)$ . We wish to show that  $\xi$  is zero. To do so, we note that  $\xi$  induces a compact and self-adjoint linear map  $K_\xi : \mathcal{H} \rightarrow \mathcal{H}$  which is uniquely determined by

$$\langle y, K_\xi z \rangle = \langle y \otimes z^*, \xi \rangle, \quad \forall y, z \in \mathcal{H}.$$

One readily checks that  $K_\xi$  intertwines the representation  $\pi$ . By the spectral theorem for compact and self-adjoint linear maps,  $\mathcal{H}$  decomposes into a direct sum of the kernel of  $K_\xi$  and finite-dimensional eigenspaces for  $K_\xi$ . Since  $\pi(g)$  commutes with  $K_\xi$  for every  $g$ , each of these finite-dimensional subspaces must be invariant under  $\pi$ . However, since  $\mathcal{H}$  is assumed to completely lack finite-dimensional sub-representations, only the kernel of  $K_\xi$  remains and we conclude that  $K_\xi$  is trivial, i.e.  $\xi$  is zero, which finishes the proof.  $\square$

#### 4. BIHARMONIC FUNCTIONS, COBOUNDARIES AND CENTRAL LIMIT THEOREMS

In this section we shall discuss a novel perspective on a powerful classical technique, often attributed to S.V. Nagaev [24], which is designed to prove central limit theorems for certain classes of Markov chains. This technique is discussed at length in the book [15], but in this paper we shall approach it in a slightly different way in the setting of random walks on groups.

We begin by describing a motivating example. Let  $(G, \mu)$  be a measured group and suppose  $d$  is a left invariant distance function on  $G$  which satisfies the moment condition

$$\int_G d(g, e)^{2+\varepsilon} d\mu(g) < \infty, \quad \text{for some } \varepsilon > 0.$$

Let  $(\Omega, \mathcal{P}) = (G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$  and if  $\omega$  is an element in  $\Omega$ , then we denote by  $\omega_n$  the  $n$ 'th coordinate of  $\omega$ . One readily checks that  $(\omega_n)$  is a sequence of independent  $\mu$ -distributed random variables on  $G$ , and we define  $(z_n)$  to be the corresponding random walk, i.e.

$$z_n(\omega) = \omega_0 \cdots \omega_{n-1}, \quad n \geq 1.$$

We note that the limit

$$\ell_d(\mu) = \lim_n \frac{1}{n} \int_{\Omega} d(z_n(\omega), e) d\mathbb{P}(\omega) = \lim_n \frac{1}{n} \int_G d(g, e) d\mu^{*n}(g)$$

exists and coincides with the drift of  $(G, \mu, d)$  defined in the first section of this paper. It follows from Theorem 1.3 that  $\ell_d(\mu)$  is positive whenever  $(G, \mu)$  is not Liouville, so in particular the drift is positive if  $G$  is non-amenable. In this case, the sequence

$$Y_n = \frac{d(z_n, e) - n\ell_d(\mu)}{\sqrt{n}}, \quad n \geq 1, \quad (4.1)$$

of random variables fluctuates around zero, and it makes sense to ask whether it has a non-trivial distributional limit.

The aim of this section is to outline a technique which isolates a class of triples  $(G, \mu, d)$  for which the sequence  $(Y_n)$  defined in 4.1 converges weakly to a non-degenerate Gaussian distribution on the real line, that is to say, we wish to impose natural conditions on  $G$ ,  $\mu$  and  $d$  such that for every continuous function  $\varphi$  on  $\mathbb{R}$  with compact support, we have

$$\lim_n \int_{\Omega} \varphi\left(\frac{d(z_n, e) - n\ell_d(\mu)}{\sqrt{n}}\right) d\mathbb{P} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-t^2/2\sigma^2} dt, \quad (4.2)$$

for some constant  $\sigma > 0$ . In probability theory, this convergence is usually denoted by  $Y_n \Rightarrow N(0, \sigma^2)$ , and we shall adopt this notation in this paper.

**4.1. Biharmonic and central limit theorems.** Let us now briefly outline how the technique of S.V. Nagaev works in this setting. Its starting point is the fundamental observation (Proposition 4.1 below) that the values of *biharmonic functions* along random walks always satisfy, under very weak assumptions, a central limit theorem. To make this observation precise, we first recall that a real-valued function  $\phi$  on  $G$  is *left Lipschitz* if the function

$$\rho_{\phi}(g) = \sup_s |\phi(sg) - \phi(s)|,$$

is finite for every  $g$  in  $G$ . One observes that if  $\phi$  is left Lipschitz, then  $\rho_{\phi}$  satisfies the triangle inequality

$$\rho_{\phi}(g_1 g_2) \leq \rho_{\phi}(g_1) + \rho_{\phi}(g_2), \quad \forall g_1, g_2 \in G,$$

and thus, if  $G$  is finitely generated, it is bounded from above by any word metric on  $G$ . Furthermore, recall that a  $\mu$ -integrable function  $\phi$  on  $G$  is *left  $\mu$ -quasiharmonic* if there exists a constant  $\ell(\phi)$  such that

$$\int_G \phi(sg) d\check{\mu}(s) = \phi(g) + \ell(\phi), \quad \forall g \in G,$$

and *right  $\mu$ -quasiharmonic* if there exists a constant  $r(\phi)$  such that

$$\int_G \phi(gs) d\mu(s) = \phi(g) + r(\phi), \quad \forall g \in G.$$

Finally, we say that  $\phi$  is *bi- $\mu$ -quasiharmonic* if it is left and right  $\mu$ -quasiharmonic. By letting  $g = e$  in the formulas above, we see that if  $\phi$  is bi- $\mu$ -quasiharmonic, then  $r(\phi) = \ell(\phi)$ .

The fundamental observation upon which the technique of S.V. Nagaev hinges can now be formulated as follows.

**Proposition 4.1.** *Let  $(G, \mu)$  be a symmetric measured group and suppose  $\phi$  is a left Lipschitz, bi- $\mu$ -quasiharmonic function on  $G$  such that*

$$\int_G \rho_{\phi}(g)^{2+\varepsilon} d\mu(g) < +\infty, \quad \text{for some } \varepsilon > 0.$$

If  $\phi$  is not identically equal to  $\ell(\phi)$ , then there exists  $\sigma > 0$  such that

$$\frac{\phi(z_n) - n\ell(\phi)}{\sqrt{n}} \Rightarrow N(0, \sigma^2).$$

*Proof.* Since  $\phi$  is a right  $\mu$ -quasiharmonic function, the sequence

$$M_n = \phi(z_n) - n\ell(\phi), \quad n \geq 1,$$

of measurable functions on  $\Omega$  forms a martingale with respect to the filtration generated by the coordinates up to  $n - 1$ . According to the martingale central limit theorem by McLeish in [23], in order to prove the distributional convergence asserted in the proposition, it suffices to show that the sequence  $(\psi_n)$  defined by

$$\psi_n(\omega) = \frac{1}{\sqrt{n}} \max \left\{ |\phi(z_{j+1}(\omega)) - \phi(z_j(\omega)) - \ell(\phi)| : j = 1, \dots, n-1 \right\}$$

is uniformly integrable and  $\int_{\Omega} \psi_n \, d\mathbb{P} \rightarrow 0$  as  $n$  tends to infinity, and

$$\lim_n \frac{1}{n} \sum_{j=1}^{n-1} |\phi(z_{j+1}(\omega)) - \phi(z_j(\omega)) - \ell(\phi)|^2 = \sigma^2, \quad (4.3)$$

almost everywhere with respect to  $\mathbb{P}$ , where  $\sigma$  is a positive constant.

Recall that by de la Vallée-Poussin Theorem,  $(\psi_n)$  is uniformly integrable if, but not only if,

$$\sup_n \int_{\Omega} |\psi_n|^{2+\varepsilon} \, d\mathbb{P} < \infty, \quad (4.4)$$

and thus to prove the two first assertions it suffices to show that

$$\int_{\Omega} |\psi_n|^{2+\varepsilon} \, d\mathbb{P} \leq \frac{1}{n^{\varepsilon/2}} \int_G \rho_{\phi}(g)^{2+\varepsilon} \, d\mu(g),$$

since the last integral is finite by assumption.

First note that the shift map  $\tau : \Omega \rightarrow \Omega$  given by  $\tau(\omega)_n = \omega_{n+1}$  preserves the probability measure  $\mathbb{P}$  on  $\Omega$  and is ergodic. Secondly, we have

$$\psi_n(\omega) \leq \frac{1}{\sqrt{n}} \cdot \max \left\{ \rho_{\phi}(\omega_j) : 1 \leq j \leq n-1 \right\}$$

for all  $n$ , so that if we define  $v(\omega) = \rho_{\phi}(\omega_0)$ , then  $v \in L^{2+\varepsilon}(\Omega, \mathbb{P})$ , and it is a straightforward exercise to show that

$$\int_{\Omega} \max_{1 \leq j \leq n-1} |v(\tau^j \omega)|^{2+\varepsilon} \, d\mathbb{P} \leq \frac{1}{n^{\varepsilon/2}} \int_{\Omega} |v(\omega)|^{2+\varepsilon} \, d\mathbb{P}(\omega) = \frac{1}{n^{\varepsilon/2}} \int_G \rho_{\phi}(g)^{2+\varepsilon} \, d\mu(g) \rightarrow 0,$$

Hence it remains to show the convergence in (4.3). For this purpose, we define the sequence

$$u_j(\omega) = |\phi(\omega_{-j} \cdots \omega_0) - \phi(\omega_{-j} \cdots \omega_{-1}) - \ell(\phi)|^2,$$

so that we can write

$$u_j(\tau^j \omega) = |\phi(z_{j+1}(\omega)) - \phi(z_j(\omega)) - \ell(\phi)|^2, \quad \forall j \geq 1.$$

We wish to prove that there exists a positive constant  $\sigma > 0$  such that

$$\sigma^2 = \lim_n \frac{1}{n} \sum_{j=1}^n u_j(\tau^j \omega)$$

almost everywhere. By Breiman's Lemma (see e.g. Lemma 14.34 in [12]), it suffices to show that

$$\int_{\Omega} \sup_j u_j \, d\mathbb{P} < \infty$$

and that there exists a function  $u \in L^1(\Omega, \mathbb{P})$  such that  $u_j \rightarrow u$  almost surely and in the  $L^1$ -norm. Indeed, if this is the case, then

$$\sigma^2 = \lim_n \frac{1}{n} \sum_{j=1}^n u_j(\tau^j \omega) = \int_{\Omega} u \, d\mathbb{P},$$

almost surely and  $\sigma = 0$  if and only if  $u$  vanishes almost everywhere. To prove the existence of a function  $u$  as above, we define the sequence

$$N_j = \phi(\omega_{-j} \cdots \omega_0) - \phi(\omega_{-j} \cdots \omega_{-1}) - \ell(\phi),$$

so that  $u_j = |N_j|^2$ , and since  $\phi$  is *left*  $\mu$ -quasiharmonic, we see that  $(N_j)$  is a martingale with respect to the filtration generated by the coordinates from  $-j$  to 0. Furthermore, since  $\phi$  is left Lipschitz, we also have that

$$C = \sup_j \int_{\Omega} |N_j|^2 \, d\mathbb{P} \leq \int_G \rho_{\phi}(g)^2 \, d\mu(g) + \ell(\phi)^2 + 2 \cdot \ell(\phi) \cdot \int_G \rho_{\phi}(g) \, d\mu(g),$$

and thus  $(N_j)$  is a  $L^2$ -bounded martingale. In particular, by the classical Martingale Convergence Theorem, there exists a function  $N_{\infty}$  in  $L^2(\Omega, \mathbb{P})$  such that  $N_j \rightarrow N_{\infty}$  almost everywhere and

$$\lim_j \int_{\Omega} |N_j - N_{\infty}|^2 \, d\mathbb{P} = 0$$

and thus, with  $u = |N_{\infty}|^2$ , we have  $u_j \rightarrow u$  almost everywhere and

$$\int_{\Omega} |u_j - u| \, d\mathbb{P} \leq \int_{\Omega} |(N_j - N_{\infty})(N_j + N_{\infty})| \, d\mathbb{P} \leq 4 \cdot C \cdot \int_{\Omega} |N_j - N_{\infty}|^2 \, d\mathbb{P} \rightarrow 0.$$

Finally, we need to show that  $u$  does not vanish almost everywhere with respect to  $\mathbb{P}$ . Note that if  $u$  vanishes almost everywhere, then so does  $N_{\infty}$  and thus

$$\lim_j \phi(\omega_{-j} \cdots \omega_0) - \phi(\omega_{-j} \cdots \omega_{-1}) = \ell(\phi)$$

almost everywhere. Hence, for any fixed  $j_0$ , by calculating the conditional expectation of the limit with respect to the  $\sigma$ -algebra generated by all coordinates strictly below  $-j_0$ , we conclude that

$$\phi(\omega_{-j_0} \cdots \omega_0) - \phi(\omega_{-j_0} \cdots \omega_{-1}) = \ell(\phi),$$

almost everywhere. In particular, since  $\mu$  is assumed to generate  $G$  as a semigroup, we have  $\phi = \ell(\phi)$  everywhere, which we have assumed is not the case.  $\square$

**4.2. Constructing bi-quasiharmonic functions.** We now return to our motivating example. As we have seen in Subsection 1.1, given any triple  $(G, \mu, d)$ , there exists a sequence  $(n_j)$  such that the limit

$$\phi(g) = \lim_{j \rightarrow \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \int_G (d(g, x) - d(x, e)) \, d\mu^{*k}(x)$$

exists for all  $g$  in  $G$ , and the function  $\phi$  satisfies

$$\int_G \phi(sg) \, d\tilde{\mu}(s) = \phi(g) + \ell_d(\mu)$$

and

$$\phi(g) \leq d(g, e) \quad \text{and} \quad \rho_{\phi}(g) = \sup_s |\phi(sg) - \phi(s)| \leq d(g, e), \quad \forall g \in G.$$

In particular,  $\phi$  is left Lipschitz and left  $\mu$ -quasiharmonic. Furthermore, if we write

$$\frac{d(z_n, e) - n\ell_d(\mu)}{\sqrt{n}} = \frac{d(z_n, e) - \phi(z_n)}{\sqrt{n}} + \frac{\phi(z_n) - n\ell_d(\mu)}{\sqrt{n}},$$

then the first term is non-negative and converges to zero in the  $L^1$ -norm if and only if

$$\lim_n \frac{1}{\sqrt{n}} \left( \int_G d(g, e) \, d\mu^{*n}(g) - n\ell_d(\mu) \right) = 0. \tag{4.5}$$

Hence, under condition 4.5, the question whether 4.2 holds is completely reduced to the question whether

$$\frac{\phi(z_n) - n\ell_d(\mu)}{\sqrt{n}} \Rightarrow N(0, \sigma^2) \quad (4.6)$$

for some positive constant  $\sigma$ .

Unfortunately, there is no reason in general to expect that  $\phi$  is also right  $\mu$ -quasiharmonic so that Proposition 4.1 can be directly applied. We approach this serious problem as follows. Let  $(B, m)$  be the Poisson boundary of  $(G, \mu)$  and note that for every  $u \in L^\infty(B)$ , the function

$$\phi_u(g) = \phi(g) + \int_G u(g^{-1}b) d\mu(b), \quad g \in G,$$

is again left Lipschitz and left  $\mu$ -quasiharmonic. Furthermore, (4.6) holds for  $\phi_u$  if and only if it holds for  $\phi$ . Hence it makes sense to ask whether we can find  $u \in L^\infty(B, m)$  such that  $\phi_u$  is right  $\mu$ -quasiharmonic. It turns out that there is a simple criterion for this. Indeed, since  $\phi$  is left Lipschitz, one can readily check that the function

$$\hat{\psi}(s) = \int_G (\phi(sg) - \phi(s)) d\mu(g),$$

is bounded and left  $\mu$ -harmonic, and thus it corresponds via the Poisson transform (discussed in the first section of this paper) to an element  $\psi$  in  $L^\infty(B)$  (which we shall refer to as the *right  $\mu$ -obstruction*), with the property that

$$\int_B \psi(b) dm(b) = \ell_d(\mu).$$

We observe that  $\phi_u$  is right  $\mu$ -quasiharmonic if and only if  $u$  satisfies the "cohomological equation"

$$u(b) - \int_B u(s^{-1}b) d\mu(s) = \psi(b) - \ell_d(\mu), \quad \text{a.e. } [m]. \quad (4.7)$$

For many triples  $(G, \mu, d)$  of interest, such as Gromov hyperbolic groups equipped with symmetric probability measures with finite exponential moments, one can show that  $\psi - \ell_d(\mu)$  must belong to a certain subspace  $\mathcal{B} \subset L^\infty(B, m)$  consisting of "smooth" functions with zero  $m$ -integrals, which admits a seminorm  $N_o$  with the property that

$$N(u) = \|u\|_\infty + N_o(u)$$

is a norm on  $\mathcal{B}$  and there exist  $0 < \tau < 1$  and an integer  $n_o$  such that the convolution operator

$$Q_\mu u(b) = \int_G u(g^{-1}b) d\mu(g)$$

satisfies the contraction bound

$$N_o(Q_\mu^{n_o} u) \leq \tau \cdot N_o(u) \quad \forall u \in \mathcal{B}. \quad (4.8)$$

Note that once such a bound has been established, it is not hard to show that the von Neumann series

$$u = \sum_{n \geq 0} Q_\mu^{*n}(\psi - \ell_d(\mu))$$

is a well-defined element in  $\mathcal{B}$  which solves the equation 4.7. The main aim of the rest of this section will be to single out a class of symmetric measured groups which comes equipped with a "natural" weakly dense semi-normed subspace of  $L^\infty(B, m)$ , which one should think of as "measurably Hölder continuous" functions, on which  $Q_\mu$  satisfies the above contraction bound.



**4.3. Besov spaces defined by product currents.** Let  $(G, \mu)$  be a countable symmetric measured group and suppose  $(X, \nu)$  is a compact  $(G, \mu)$ -space, that is to say,  $X$  is a compact metrizable space equipped with an action of  $G$  by homeomorphisms such that  $\nu$  satisfies the equation

$$\int_G \int_X \phi(s^{-1}x) d\nu(s) d\mu(s) = \int_X \phi(x) d\nu(x)$$

for all  $\phi \in C(X)$ . If  $\nu$  is non-atomic, then we can think of the product measure  $\nu \otimes \nu$  as a probability measure on the (in general) *non-compact* space  $\partial^2 X = X \times X \setminus \Delta X$ , where  $\Delta X$  denotes the (closed) diagonal subspace in  $X \times X$ . A non-negative Borel measurable function  $\rho$  on  $\partial^2 X$  is called a *product current* if the (possibly infinite) Borel measure  $\eta$  on  $\partial^2 X$  defined by

$$\int_{\partial^2 X} \phi(x, y) d\eta(x, y) = \int_{\partial^2 X} \phi(x, y) \rho(x, y) d\nu(x) d\nu(y)$$

is invariant with respect to the diagonal action of  $G$  on  $\partial^2 X \subset X \times X$ . One can readily check that this condition simply translates to the validity of the equation

$$\rho(gx, gy) \sigma_\nu(g, x) \sigma_\nu(g, y) = \rho(x, y) \quad (4.9)$$

for all  $g$  in  $G$  and for almost every  $(x, y)$  with respect to the product measure  $\nu \otimes \nu$ .

We stress that not every  $(G, \mu)$ -space admits a product current. However, certain classes of countable groups, such as Gromov hyperbolic groups and lattices in higher rank Lie groups, carry symmetric probability measures with the property that their Poisson boundaries (in some compact model) admit "natural" and "geometrically defined" product currents. We refer the reader to Section 5 of the paper [6] for a detailed discussion about product currents for Gromov hyperbolic measured groups. In this case,  $X$  is the Gromov boundary of the hyperbolic group  $G$ , equipped with a certain distance function  $d_o$ , and  $\rho$  is roughly proportional to  $d_o(x, y)^{-D}$ , where  $D$  is a constant related to the Hausdorff dimension of  $X$ .

Given a product current  $\rho$  for a  $(G, \mu)$ -space  $(X, \nu)$  and given  $\varepsilon > 0$ , we define a semi-norm  $N_{\rho, \varepsilon}$  on a subspace  $\mathcal{B}_{\rho, \varepsilon} \subset L^\infty(X, \nu)$  by

$$N_{\rho, \varepsilon}(u) = \int_{\partial^2 X} |u(x) - u(y)| \rho(x, y)^{\frac{1}{2} + \varepsilon} d\nu(x) d\nu(y),$$

where  $\mathcal{B}_{\rho, \varepsilon}$  consists of those elements in  $L^\infty(X, \nu)$  with finite  $N_{\rho, \varepsilon}$ -seminorms. We shall refer to linear space  $(\mathcal{B}_{\rho, \varepsilon}, N_{\rho, \varepsilon})$  as the *Besov space associated to  $\rho$  of order  $\varepsilon$* , and since  $\rho$  usually blows up close to the diagonal, we may think of  $\mathcal{B}_{\rho, \varepsilon}$  as a "measurable" replacement of Hölder continuous functions on  $X$ . As the following proposition will show, there is a simple criterion for the validity of the contraction bound 4.8 for  $Q_\mu$  acting on the space  $\mathcal{B}_{\rho, \varepsilon}$ .

**Proposition 4.2.** *Let  $(G, \mu)$  be a countable measured group and suppose  $(X, \nu)$  is a  $(G, \mu)$ -space which admits a product current  $\rho$ . Given  $\varepsilon > 0$  and an integer  $n$ , we define*

$$\tau_{\varepsilon, n} = \text{ess sup} \int_G \sigma_\nu(g, \cdot)^{1-2\varepsilon} d\mu^{*n}(g).$$

*Then  $N_{\rho, \varepsilon}(Q_\mu^n u) \leq \tau_{\varepsilon, n} \cdot N_{\rho, \varepsilon}(u)$  for all  $u \in \mathcal{B}_{\rho, \varepsilon}$ .*

*Proof.* First recall that

$$\rho(sx, sy) \sigma_\nu(s, x) \sigma_\nu(s, y) = \rho(x, y)$$

for almost every  $(x, y)$  with respect to  $\nu \otimes \nu$ . Hence, we have

$$\begin{aligned}
N_{\rho, \varepsilon}(Q_{\mu}^{*n}u) &\leq \int_G \int_{\partial^2 X} |u(s^{-1}x) - u(s^{-1}y)| \rho(x, y)^{\frac{1}{2} + \varepsilon} d\nu(x) d\nu(y) d\mu^{*n}(s) \\
&= \int_G \int_{\partial^2 X} |u(x) - u(y)| \rho(sx, sy)^{\frac{1}{2} + \varepsilon} \sigma_{\nu}(s, x) \sigma_{\nu}(s, y) d\nu(x) d\nu(y) d\mu^{*n}(s) \\
&= \int_G \int_{\partial^2 X} |u(x) - u(y)| \rho(x, y)^{\frac{1}{2} + \varepsilon} \sigma_{\nu}(s, x)^{\frac{1}{2} - \varepsilon} \sigma_{\nu}(s, y)^{\frac{1}{2} - \varepsilon} d\nu(x) d\nu(y) d\mu^{*n}(s) \\
&\leq \left( \text{ess sup} \int_G \sigma_{\nu}(s, \cdot)^{\frac{1}{2} - 2\varepsilon} d\mu^{*n}(s) \right) \cdot N_{\rho, \varepsilon}(u),
\end{aligned}$$

where we in the last line used Hölder's inequality twice.  $\square$

**4.4. Minimality and non-invariance force contraction.** The aim of the final subsection of this section will be to isolate natural conditions on a compact  $(G, \mu)$ -space  $(X, \nu)$  which will force the existence of an integer  $n$ , for every given  $\varepsilon > 0$ , such that

$$\tau_{n, \varepsilon} = \text{ess sup} \int_G \sigma_{\nu}(s, \cdot)^{1 - 2\varepsilon} d\mu^{*n}(s) < 1. \quad (4.10)$$

We shall henceforth assume that the functions  $x \mapsto \sigma_{\nu}(s, x)$  are continuous for every  $s$  in  $G$ . Although this assumption is not absolutely necessary, it will simplify many of the arguments below. Furthermore, we may without loss of generality assume that the identity belongs to the support of  $\mu$ . Indeed, if not, then we can replace  $\mu$  with the probability measure

$$\mu_o = \frac{1}{2}\delta_e + \frac{1}{2}\mu,$$

with respect to which  $\nu$  is still stationary, and (4.10) holds for  $\mu$  if and only if it holds for  $\mu_o$ . Note that the supports of  $\mu_o^{*n}$  forms an *increasing* family of sets in  $G$  which asymptotically exhausts  $G$ .

First recall that by (1.3) (which holds for every  $(G, \mu)$ -space), we have

$$\int_G \sigma_{\nu}(s, x) d\mu^{*n}(s) = 1$$

for all  $n$  and for almost every  $x$  in  $X$ . In particular,  $\tau_{n, \varepsilon}$  is always bounded by one for all  $n$  and  $\varepsilon$ , and (4.10) fails if and only if for every  $n$ , there exists  $x_n \in X$  such that

$$\int_G \sigma_{\nu}(s, x_n)^{1 - 2\varepsilon} d\mu^{*n}(s) = 1.$$

In other words, for every  $n$ , we have equality in Hölder's inequality (when integrating against  $\mu^{*n}$ ), which clearly forces the identities

$$\sigma_{\nu}(s, x_n) = 1 \quad \forall s \in \text{supp } \mu^{*n}$$

for all  $n$ . Let  $x_{\infty}$  be an accumulation point of the sequence  $(x_n)$  in  $X$ . Since  $\sigma_{\nu}(s, \cdot)$  is continuous for every  $s$  and the supports of  $\mu^{*n}$  is an increasing exhausting family of sets in  $G$ , we conclude that

$$\sigma_{\nu}(s, x_{\infty}) = 1, \quad \forall s \in G.$$

Let us now further assume that the  $G$ -action on  $X$  is *minimal*, i.e. every  $G$ -orbit is dense. Then, by the cocycle equation (1.2), which holds for every  $(G, \mu)$ -space, we have  $\sigma_{\nu}(s, tx_{\infty}) = 1$  for all  $s, t$  in  $G$ , and since  $Gx_{\infty}$  is dense and  $\sigma_{\nu}(s, \cdot)$  is continuous, we conclude that  $\sigma_{\nu}(s, x) = 1$  for all  $s$  in  $G$  and  $x$  in  $X$ , or equivalently,  $\nu$  is  $G$ -invariant. We summarize the above discussion in the following proposition.

**Proposition 4.3.** *Let  $(G, \mu)$  be a countable measured group and suppose  $(X, \nu)$  is a compact minimal  $(G, \mu)$ -space such that  $\sigma_\nu(s, \cdot)$  is continuous for every  $s$  in  $G$ . If  $\nu$  is not  $G$ -invariant, then for every  $\varepsilon > 0$ , there exists an integer  $n$  such that*

$$\sup \int_G \sigma_\nu(s, \cdot)^{1-2\varepsilon} d\mu^{*n}(s) < 1.$$

**Remark 4.1.** The assumptions in the last proposition are satisfied for every symmetric probability measure  $\mu$  with finite exponential moments (with respect to the any word metric) on any non-elementary Gromov hyperbolic group, where  $(X, \nu)$  denotes its Gromov boundary and  $\nu$  is the unique  $\mu$ -stationary measure on  $X$ . Hence Proposition 4.3 gives a new proof of the main technical estimate in the author's paper [4].

Although Proposition 4.3 assumes a lot about the topological and dynamical structure of  $(X, \nu)$ , there is no assumption about the moments of  $\mu$ . In particular, Proposition 4.3, as well as the discussions about product currents proceeding it, also apply to the Furstenberg boundary action of a lattice  $G$  in a simple Lie group  $H$ , at least when the  $\mu$ -stationary measure  $\nu$  on the Furstenberg boundary  $H/P$  (here  $P$  is a minimal parabolic subgroup of  $H$ ) belong to the Haar measure class. Such probability measures on the lattice always exist (see e.g. [9]), but they tend to have very heavy tails. Since the Furstenberg boundary of a simple Lie group, equipped with the Haar measure, always admits a product current (upon identifying a conull subset of  $H/P \times H/P$  with  $H/A$ , where  $A$  is the (unimodular) split torus of  $G$ ), Proposition 4.3 in particular implies that every function in the associated Besov space  $\mathcal{B}_{\rho, \varepsilon}$  is in fact of the form  $\phi - \mu * \phi$  for some  $\phi \in \mathcal{B}_{\rho, \varepsilon}$ .

## 5. PRODUCT SETS IN GROUPS

This final section is concerned with the structure of difference sets in free groups, and the aim here is to give a short and rather elementary proof of a weaker version of a recent theorem by the author and A. Fish (Theorem 1.1 in [5]). We begin by providing some background and motivation.

A significant part of additive combinatorics is concerned with special instances of the following phenomenology: If  $G$  is a countable group and  $A, B \subset G$  are "large" subsets, then the product set  $AB$  should exhibit "substantial sub-structures". The exact meanings of these notions varies a lot depending on the context, and in this section we shall only be concerned with (partially) extending the following result by Khintchine [21] and Følner [11], which was one of the first observations of this phenomenology (at least in the setting of discrete groups).

**Theorem 5.1.** *Suppose  $A_1, \dots, A_k \subset \mathbb{Z}$  are subsets which are "large" in the sense that*

$$\overline{\lim}_{n \rightarrow \infty} \frac{|A_i \cap [-n, n]|}{2n+1} > 0, \quad \forall i = 1, \dots, k.$$

*Then their difference sets contain "substantial sub-structures", in the sense that there exists a finite set  $F \subset \mathbb{Z}$  such that*

$$F + \bigcap_{i=1}^k (A_i - A_i) = \mathbb{Z}.$$

The additive group of integers is of course nothing but the free group on one generator. A first naive attempt to extend Theorem 5.1 to free group on two or more generators could be devised along the following lines. Let  $\mathbb{F}_2$  denote the free group on two (free) generators  $a$  and  $b$  and let  $B_n$  denote the ball of radius  $n$  with respect to these generators, that is to say,  $B_n$  consists of all the words in  $a$  and  $b$  and their inverses whose reduced form have length at most  $n$ . In analogy with Theorem 5.1 (where the "balls" with respect to the one free generator 1 are simply given by the interval  $[-n, n]$ ), we say that a set  $A \subset \mathbb{F}_2$  is *upper large* if

$$\overline{\lim}_{n \rightarrow \infty} \frac{|A \cap B_n|}{|B_n|} > 0.$$

However, we warn the reader that upper large sets could be *very* sparse in  $\mathbb{F}_2$ ; for instance, given any increasing sequence  $(r_i)$  of positive integers, the set

$$A = \bigcup_{i=1}^{\infty} (B_{r_{i+1}} \setminus B_{r_i}) \subset \mathbb{F}_2 \quad (5.1)$$

satisfies

$$\overline{\lim}_{n \rightarrow \infty} \frac{|A \cap B_n|}{|B_n|} \geq \frac{2}{3}.$$

We do not expect to say anything intelligent about differences of sets like these, so we slightly modify our notion of largeness to exclude too sparse examples. Define the *sphere*  $S_n$  of radius  $n$  by  $S_n = B_n \setminus B_{n-1}$  and say that a set  $A \subset G$  is *large* if

$$\overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \frac{|A \cap S_n|}{|S_n|} > 0.$$

We see that for a set as in (5.1) to be large in this sense, serious growth constraints on the sequence  $(r_i)$  have to be imposed, so the notion of largeness is *strictly* weaker than upper largeness.

One can now ask whether something like Theorem 5.1 could be true for large sets. However, already simple considerations show that great care has to be taken to even formulate the right statement. Indeed, it is not hard to construct (and we refer to [5] for details) large subsets  $A_1, A_2, A_3 \subset \mathbb{F}_2$  such that

$$A_1 A_1^{-1} \cap A_2 A_2^{-1} \cap A_3 A_3^{-1} = \{0\}$$

and for which there is no finite subset  $F \subset G$  such that  $FA_i A_i^{-1} = \mathbb{F}_2$  for *some*  $i = 1, 2, 3$ . However, the situation is not completely hopeless if one is willing to slightly weaken the notion of "substantial sub-structure" as the following recent observation (see Corollary 1.2 in [5]) by the author and A. Fish shows.

**Theorem 5.2** (Björklund-Fish, weak version). *Suppose  $A \subset \mathbb{F}_2$  is "large" in the sense that*

$$\overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \frac{|A \cap S_n|}{|S_n|} > 0.$$

*Then there exists a finite set  $F \subset \mathbb{F}_2$  such that  $FAA^{-1}$  contains a right translate of every finite subset of  $G$ .*

We stress that this is not the formulation of Corollary 1.2. in [5], so we first take a moment to rewrite Theorem 5.2 in a language which better align with the present paper (and with [5]). Let  $G = \mathbb{F}_2$  and define the probability measures  $(\sigma_n)$  on  $G$  (uniform sphere averages) by

$$\sigma_0 = \delta_e \quad \text{and} \quad \sigma_n = \frac{1}{|S_n|} \sum_{s \in S_n} \delta_s, \quad \text{for } n \geq 1.$$

It is not hard to verify the relations

$$\sigma_n * \sigma_1 = \frac{3}{4} \cdot \sigma_{n+1} + \frac{1}{4} \cdot \sigma_{n-1}, \quad \forall n \geq 1, \quad (5.2)$$

which in particular shows that every  $\sigma_n$  can be written as a convex combination of convolution powers of  $\sigma_1$  and  $\delta_e$ .

Let  $M(G)$  denote the convex set of all means on  $G$ , i.e. the set of all linear functionals on  $\ell^\infty(G)$  which are positive and unital (i.e.  $\lambda(1) = 1$ ). We note that every mean  $\lambda$  gives rise to a *finitely additive* probability measure  $\lambda'$  on  $G$  via the formula

$$\lambda'(C) = \lambda(\chi_C), \quad C \subset G,$$

and by the Banach-Alaoglo's Theorem, the set  $M(G)$  is compact with respect to the weak\*-topology. In particular, every sequence  $(\lambda_i)$  of the form

$$\lambda_i = \frac{1}{m_i} \sum_{n=1}^{m_i} \sigma_n, \quad i \geq 1,$$

for some increasing sequence  $(m_i)$ , must have at least one cluster point  $\lambda$ , which by the relations in (5.2) is necessarily left  $\sigma_1$ -harmonic (note that  $\sigma_1$  is symmetric), that is to say

$$\int_G g \cdot \lambda(\varphi) d\sigma_1(g) = \lambda(\varphi), \quad \forall \varphi \in \ell^\infty(G),$$

where  $G$  acts on  $\ell^\infty(G)$  (and hence on its dual via the transpose map) by the left regular representation. In particular, if we choose a sequence  $(m_i)$  such that

$$\lim_{i \rightarrow \infty} \frac{1}{m_i} \sum_{n=1}^{m_i} \frac{|A \cap S_n|}{|S_n|} > 0,$$

and a cluster point  $\lambda$  of the corresponding sequence of means as above, then  $\lambda'(A) > 0$ . The aim is now to show that this condition automatically forces the existence of a finite set  $F \subset G$  such that  $FAA^{-1}$  contains a right translate of every finite subset of  $G$ .

It will be convenient to adopt a slightly more general perspective on these matters. Let  $(G, \mu)$  be a countable symmetric measured group. We say that an element  $\lambda \in M(G)$  is *left  $\mu$ -harmonic* if

$$\int_G \lambda(\varphi(g^{-1} \cdot)) d\mu(g) = \lambda(\varphi), \quad \forall \varphi \in \ell^\infty(G).$$

We say that a set  $T \subset G$  is *right thick* if it contains a right translate of every finite subset of  $G$ , that is to say, if for every finite subset  $F \subset G$ , there exists  $g \in G$  such that  $Fg \subset T$ . It is not hard to see that a set  $T \subset G$  is right thick if and only if for every finite set  $F \subset G$ , the intersection of all left translates of the form  $fT$ , with  $f \in F$ , is non-empty. In particular, if  $\lambda$  is a left  $\mu$ -harmonic mean on  $G$  such that  $\lambda'(T) = 1$ , then

$$\int_G \lambda'(gT) d\mu^{*k}(g) = \lambda'(T) = 1, \quad \forall k \geq 1,$$

which shows that  $\lambda'(gT) = 1$  for all  $g \in G$ . Since  $\lambda'$  is a finitely additive measure, we conclude that for every finite set  $F \subset G$ , the intersection of all left translates  $fT$ , with  $f \in F$ , still has full  $\lambda'$ -measure (so in particular it is non-empty), which shows that  $T$  must be right thick.

Theorem 5.2 will now follow from the following proposition.

**Proposition 5.1.** *Let  $(G, \mu)$  be a measured group and suppose  $A \subset G$  has positive measure with respect to some left  $\mu$ -harmonic mean on  $G$ . Then there exists a finite set  $F \subset A$  such that  $FAA^{-1}$  has measure one with respect to some left  $\mu$ -harmonic mean on  $G$ .*

To prove this proposition, we will need the following result, which is not hard, and follows from quite standard correspondence principles. However, the author is not aware of a (short) proof which avoids various technical manipulations with extreme points in the simplex of  $\mu$ -harmonic measures on compact  $G$ -spaces. We shall therefore omit the proof, and refer the interested reader to Proposition 1.2 in [5], where a much stronger result is proven.

**Lemma 5.1.** *Fix  $\varepsilon > 0$  and suppose  $A \subset G$  has positive measure with respect to some left  $\mu$ -harmonic mean. Then there exists a finite set  $F \subset G$  and a (possibly different) left  $\mu$ -harmonic mean  $\eta$  on  $G$  such that  $\eta(FA) \geq 1 - \varepsilon$ .*

If one is willing to take this lemma for granted, then we argue as follows. Suppose  $A \subset G$  has positive  $\lambda'$ -measure for some left  $\mu$ -harmonic mean  $\lambda$  on  $G$ . Fix  $\varepsilon > 0$  and find, by the previous lemma, a finite set  $F \subset G$  and a left  $\mu$ -harmonic mean  $\eta$  on  $G$  such that

$$\eta(FA) \geq 1 - \varepsilon \cdot \eta(A).$$

We note that

$$FAA^{-1} \supset \{g \in G : \eta(FA \cap gA) > 0\} \supset \{g \in G : \eta(gA) > \varepsilon \cdot \eta(A)\},$$

and the function

$$u(g) = \eta(gA) - \varepsilon \cdot \eta(A), \quad g \in G,$$

is a real-valued bounded left  $\mu$ -harmonic function on  $G$ . Furthermore, if  $0 < \varepsilon < 1$ , then  $u(e) > 0$  and  $u$  is *positively correlated* in the sense that

$$\|u\|_\infty = \sup \{u(g) : g \in G\},$$

so Proposition 5.1 will follow from the "zero-one law" stated below.

**Lemma 5.2.** *Let  $(G, \mu)$  be a measured group and suppose  $u$  is a bounded real-valued left  $\mu$ -harmonic function on  $G$ . Define the set*

$$S_u = \{g \in G : u(g) > 0\} \subset G.$$

*If  $u$  is positively correlated, then there exists a left  $\mu$ -harmonic mean which gives measure one to the set  $S_u$ .*

To prove this lemma, we first note that for any mean  $\lambda$  on  $G$ , for any  $\varepsilon > 0$  and for every bounded function  $u$  on  $G$ , we have

$$\begin{aligned} \lambda'(\{g \in G : u(g) > 0\}) &\geq \lambda'(\{g \in G : u(g) \geq (1 - \varepsilon) \cdot \|u\|_\infty\}) \\ &= 1 - \lambda'(\{g \in G : u(g) < (1 - \varepsilon) \cdot \|u\|_\infty\}) \\ &= 1 - \lambda'(\{g \in G : \|u\|_\infty - u(g) > \varepsilon \cdot \|u\|_\infty\}) \\ &\geq 1 - \frac{1}{\varepsilon \cdot \|u\|_\infty} \cdot (\|u\|_\infty - \lambda(u)), \end{aligned}$$

by Chebyshev's inequality (which works equally well for finitely additive probability measures). Hence it suffices to show that whenever  $u$  is a positively correlated left  $\mu$ -harmonic function, there exists a left  $\mu$ -harmonic mean  $\lambda$  such that  $\lambda(u) = \|u\|_\infty$ . To prove this, we fix a sequence  $(g_n)$  such that

$$\lim_n u(g_n) = \sup \{u(g) : g \in G\},$$

and define the sequence  $(\lambda_m)$  of means on  $G$  by

$$\lambda_m(\phi) = \frac{1}{m} \sum_{n=1}^m \int_G \phi(xg_n) d\check{\mu}^{*n}(x), \quad \phi \in \ell^\infty(G).$$

Since  $u$  is left  $\mu$ -harmonic, we have  $\lambda_m(u) = u(g_m)$  for all  $m$ , and one readily checks that any cluster point  $\lambda$  of the sequence  $(\lambda_m)$  in  $M(G)$  is left  $\mu$ -harmonic and satisfies  $\lambda(u) = \lim_m u(g_m)$ .

## 6. APPENDIX I: HARMONIC FUNCTIONS AND AFFINE ISOMETRIC ACTIONS ON HILBERT SPACES

As part of Theorem 1.3, we proved that if  $(G, \mu)$  is a measured Liouville group and  $u$  is a left Lipschitz and left (quasi-) $\mu$ -harmonic function on  $G$ , then  $u$  must be a homomorphism. The aim of this appendix is to show that the combination "LEFT Lipschitz" and "LEFT quasi- $\mu$ -harmonic" is crucial, and if one (but not both) is replaced by a "RIGHT", then the situation is quite different. Indeed, we shall prove the following theorem, whose origin is hard to track down, but which is well-known to experts.

**Theorem 6.1** (Folklore). *Every infinite, finitely generated and symmetric measured group  $(G, \mu)$ , where  $\mu$  is assumed to be finitely supported, admits a non-trivial left Lipschitz and right  $\mu$ -harmonic function.*

Since every *non-amenable* measured group  $(G, \mu)$  admits a wealth of non-trivial *bounded* right  $\mu$ -harmonic functions, the theorem is perhaps most interesting for amenable groups. However, the construction which we will describe below works for a larger class of groups, namely those which admit affine isometric actions on (real) Hilbert spaces with *unbounded* orbits. It is well-known (see e.g. Theorem 13.10 in [12]) that this is equivalent to assuming that the group  $G$  does *not* have Kazhdan's Property (T). In particular, our construction will work for every countable (infinite) *amenable* group.

Recall that if  $\mathcal{H}$  is a real Hilbert space, then a map  $T : \mathcal{H} \rightarrow \mathcal{H}$  is an *affine isometry*  $T$  if it can be written on the form

$$Tx = Ux + b, \quad x \in \mathcal{H},$$

for some *linear* isometry  $U$  of  $\mathcal{H}$  and  $b \in \mathcal{H}$ . Clearly, the set of affine isometries of  $\mathcal{H}$  forms a group  $\text{Aff}(\mathcal{H})$  under composition, and a homomorphism  $\alpha : G \rightarrow \text{Aff}(\mathcal{H})$  is called an *affine isometric action* of  $G$  on  $\mathcal{H}$ . Explicitly, we have

$$\alpha(g)x = \pi(g)x + b(g)$$

for some linear isometric representation  $\pi$  of  $G$  and a map  $b : G \rightarrow \mathcal{H}$  which satisfies

$$b(gh) = b(g) + \pi(g)b(h), \quad \forall g, h \in G.$$

We shall refer to such maps as  $\pi$ -*cocycles*, and we note that the action  $\alpha$  has bounded orbits if and only if the corresponding  $b$  is a norm-bounded function on  $G$ .

**Proposition 6.1.** *Let  $(G, \mu)$  be a finitely generated and symmetric measured group, where  $\mu$  is assumed to be finitely supported, and suppose  $\alpha$  is an affine isometric action of  $G$  on a real Hilbert space  $\mathcal{H}$  without unbounded orbits. Then there exists  $x_o$  and  $y$  in  $\mathcal{H}$  such that*

$$f(g) = \langle y, \alpha(g) \cdot x_o \rangle_{\mathcal{H}}, \quad g \in G,$$

*is an unbounded, left Lipschitz and right  $\mu$ -harmonic function on  $G$ .*

*Sketch of the proof.* Recall that  $\alpha$  can be written on the form

$$\alpha(g)x = \pi(g)x + b(g),$$

for some linear isometric representation  $\pi$  of  $G$  and a  $\pi$ -cocycle  $b : G \rightarrow \mathcal{H}$ . The assumption that the  $\alpha$  has unbounded orbits simply means that

$$\sup_{g \in G} \|\alpha(g)x\| = \infty, \quad \forall x \in \mathcal{H},$$

and we shall prove that there exists  $x_o \in \mathcal{H}$  such that the orbit map

$$F(g) = \alpha(g)x_o, \quad g \in G,$$

satisfies  $F * \mu = F$  in  $\mathcal{H}$ , or equivalently (after some easy manipulations)

$$\int_G (x_o - \alpha(s)x_o) d\mu(s) = 0. \tag{6.1}$$

By the uniform boundedness principle, if

$$\sup_{g \in G} |\langle y, \alpha(g)x_o \rangle| < +\infty$$

for all  $y \in \mathcal{H}$ , then

$$\sup_{g \in G} \|\alpha(g)x_o\| < +\infty,$$

which is a contradiction, and we conclude that there must exist  $y \in \mathcal{H}$  such that the function

$$f(g) = \langle y, \alpha(g)x_o \rangle, \quad g \in G,$$

is an *unbounded* (and hence non-constant) right  $\mu$ -harmonic function on  $G$ . Also note that

$$|f(sg) - f(s)| = |\langle y, \pi(s)(\pi(g)x_o - x_o + b(g)) \rangle| \leq \|y\| \cdot (2\|x_o\| + \|b(g)\|),$$

for all  $g$  and  $s$ , which shows that  $f$  is left Lipschitz.

To establish the existence of  $x_o \in \mathcal{H}$  such that (6.1) holds, we argue as follows. Consider the "energy functional"

$$E(x) = \int_G \|\alpha(s)x - x\|^2 d\mu(s), \quad x \in \mathcal{H},$$

which is well-defined since  $\mu$  is finitely supported (but clearly this assumption can be substantially weakened). One readily checks that  $E$  admits a local minimum  $x_o$ , and thus

$$\left. \frac{d}{dt} E(x_o + tv) \right|_{t=0} = 0, \quad \forall v \in \mathcal{H}.$$

The left hand side can be easily calculated. Indeed, after a series of calculations, using the assumptions that  $\mathcal{H}$  is a real Hilbert space and  $\mu$  is a symmetric measure on  $G$ , we arrive at the identities,

$$\left. \frac{d}{dt} E(x_o + tv) \right|_{t=0} = 4 \cdot \langle v, \int_G (x_o - \alpha(s)x_o) d\mu(s) \rangle = 0,$$

for all  $v \in \mathcal{H}$ , from which (6.1) follows.  $\square$

## 7. APPENDIX II: OPEN PROBLEMS AND REMARKS

We collect in this appendix some questions and remarks relating to the topics discussed in this paper.

**7.1. Drifts of random walks on homogeneous spaces.** Let  $(G, \mu)$  be a countable measured group and denote by  $(B, m)$  its Poisson boundary. Clearly, if  $H < G$  is a subgroup which acts ergodically on  $(B, m)$ , then there are no non-constant *bounded* left  $\mu$ -harmonic functions on the quotient space  $G/H$ . However, it certainly also makes sense to ask whether *unbounded* left  $\mu$ -harmonic functions can exist on the quotient space  $G/H$ , at least when  $H$  has infinite index in the group  $G$ .

For instance, in the extreme case when  $\mu$  is symmetric and finitely supported such that  $(G, \mu)$  is a Liouville group (that is to say,  $(B, m)$  is just a singleton space) and  $H$  is the trivial subgroup, then the construction in Appendix I, shows that there are always unbounded left  $\mu$ -harmonic functions.

A less extreme case is suggested by Corollary 2.1, which can be equivalently stated as the assertion that there are no non-constant bounded left  $\mu_o \otimes \check{\mu}_o$ -harmonic functions on the quotient  $G_o \times G_o / \Delta_2 G_o$  for any measured group  $(G_o, \mu_o)$ . In this setting, the problem above can be equivalently formulated as follows.

**Problem 1.** Does every measured group  $(G, \mu)$  admit a non-trivial bi- $\mu$ -harmonic (left and right  $\mu$ -harmonic) function?

The problem for general quotient spaces seems intractable, and there could very well be obvious counter-examples.

**Problem 2.** Construct a countable measured group  $(G, \mu)$  and an infinite index subgroup  $H < G$  such that the quotient space  $G/H$  does not admit *any* non-constant left (quasi-) $\mu$ -harmonic functions whatsoever.

It is clear that the notion of drift can be generalized to invariant metrics on more general  $G$ -spaces (in particular coset spaces). An affirmative answer to the following question would generalize the Karlsson-Ledrappier Theorem (Theorem 1.3) to this setting.



**Problem 3.** Let  $(G, \mu)$  be a measured group and  $H < G$  a subgroup which acts ergodically on the Poisson boundary of  $(G, \mu)$ . If  $d$  is a left  $G$ -invariant metric on the quotient space  $G/H$ , is it then true that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \int_G d(gH, H) d\mu^{*k}(g) = 0?$$

One could start by analyzing the following special case which corresponds to the case when  $G = G_o \times G_o$  and  $H = \Delta_2 G_o$  and  $\mu = \mu_o \otimes \mu_o$ , for some countable group  $G_o$  and some symmetric measure  $\mu_o$  on  $G_o$ .

**Problem 4.** Let  $(G, \mu)$  be a symmetric measured group and suppose there exists a *bi-invariant* (conjugation-invariant) and  $\mu$ -integrable (semi-)metric  $d$  on  $G$ . Is  $\ell_d(\mu) = 0$ ?

For instance, as a first test case, one could focus on the commutator subgroup  $G$  of the free group on two generators and the stable commutator length on  $G$ .

**7.2. Harmonic Kronecker factors.** Let  $G$  be a countable group and  $(X, \nu)$  a non-singular ergodic  $G$ -space. Let  $\mathcal{K}$  denote the smallest  $G$ -invariant sub- $\sigma$ -algebra of the Borel  $\sigma$ -algebra on  $X$  with the property that  $\mathcal{K} \times \mathcal{K}$  contains the  $\sigma$ -algebra of all  $G$ -invariant subsets in  $X \times X$ . When  $\nu$  is  $G$ -invariant, this  $G$ -invariant  $\sigma$ -algebra (or its corresponding factor) is usually called the *Kronecker factor*, and it is a classical fact that the factor  $G$ -space is isomorphic to an action by rotations of  $G$  on a compact homogeneous space. Except for some remarks in [2], this factor does not seem to have attracted much attention, and it seems hard to say anything significant about it in this generality. However, it could be that the situation for  $(G, \mu)$ -spaces is more amenable for a closer analysis.

**Problem 5.** Let  $(G, \mu)$  be a symmetric measured group with Poisson boundary  $(B, m)$  and suppose  $(X, \nu)$  is an ergodic  $(G, \mu)$ -space which admits  $(B, m)$  as a factor. Assume that the product of  $(X, \nu)$  with itself is *not* ergodic. Does this mean that there exists a factor  $(Y, \eta)$  of  $(X, \nu)$  which is a non-trivial isometric extension of  $(B, m)$ ?

Put differently, is  $(Y, \eta)$  isomorphic (as a  $G$ -space) to a skew product of the form  $(B \times K/K_o, m \otimes \eta)$ , where  $K$  is a compact group and  $K_o$  a closed subgroup and  $\eta$  is the Haar probability measure on  $K/K_o$ , such that the  $G$ -action can be written as

$$g(b, z) = (gb, c(g, b)z), \quad (b, z) \in B \times K/K_o,$$

where  $c : G \times B \rightarrow K$  is a measurable cocycle?

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